STAT 511: Summer 2018 – Quiz 3

Name: Class Time:

1. (4 points) A coin with P(Head) = 1/3 is flipped 6 times. Let X and Y be the number of Heads and Tails observed in these 6 flips respectively. Find out Cov(X,Y) and $\rho_{X,Y}$. (Hint: Consider the variance of X+Y.)

Clearly, $X \sim Bin(6, \frac{1}{3})$ and $Y = 6 - X \sim Bin(6, \frac{2}{3})$

$$\Rightarrow V(X) = 6 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{3} = V(Y)$$

As $X + Y = 6$ with probability 1, $V(X+Y) = 0$

$$\Rightarrow V(X) + V(Y) + 2\,Cov(X,Y) = 0 \Rightarrow Cov(X,Y) = -\frac{V(X)+V(Y)}{2} = -V(X) = -\frac{4}{3}$$

Now, $\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-4/3}{\sqrt{4/3 \times 4/3}} = -1$ (Alternatively, $Cov(X,Y) = Cov(X, 6-X) = -V(X) = -\frac{4}{3}$ and $\rho_{X,Y} = \rho_{X,6-X} = -1$)

2. (4 points) Let X and Y be i.i.d. N(0,1) random variables and Z = 1 + X and W = XY^2 + X – 2. Find Cov(Z,W).

$$\begin{aligned}
Cov(Z,W) &= Cov(1+X, XY^2+X-2) \\
&= Cov(X, XY^2+X) \\
&= Cov(X, XY^2) + Cov(X,X) \\
&= E(X^2Y^2) - E(X)\cdot E(XY^2) + V(X) \\
&= E(X^2)E(Y^2) - E(X)E(X)\cdot E(Y^2) + V(X) \\
&= 1 \times 1 - 0 \times 0 \times 1 + 1 = 2 \quad (\text{as } X, Y \overset{iid}{\sim} N(0,1))
\end{aligned}$$

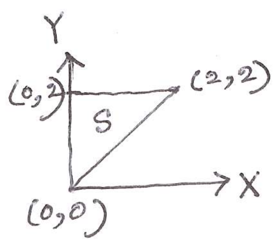
3. (4 points) Let the joint PDF of (X,Y) be

$$f(x,y) = \begin{cases} cx^m y^n & if\ (x,y) \in S \\ 0 & otherwise \end{cases}$$

where S = { (x,y) : 0 ≤ x ≤ y ≤ 2 } and c, m and n are constants.

(a) (1 point) Are X and Y independent? Write a brief explanation.

The joint support S is triangular, so X and Y are NOT independent.

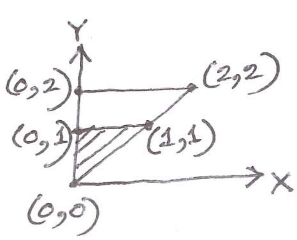


(b) (1 point) If m = n = 0, find the value of c.

If $m = n = 0$, $f(x,y) = c\, I_S(x,y)$

So (X,Y) is jointly Uniform on $S \Rightarrow c = \frac{1}{|S|} = \frac{1}{\frac{1}{2} \times 2 \times 2} = \frac{1}{2}$.

(c) (2 points) If m = n = 0, find P(Y < 1). (Hint: You don't need to perform any integration for parts b and c)



$$P(Y<1) = \frac{\text{area of the shaded triangle}}{|S|} = \frac{\frac{1}{2} \times 1 \times 1}{\frac{1}{2} \times 2 \times 2} = \frac{1}{4}$$